Approximation of Analog Controllers for Sampled-Data Systems

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I. Introduction

FOR a sampled data feedback system, the compensator consists of a prefilter $F(s)$, a sampler with sample period T, a digital computer $D(z)$, and a hold device $H(s)$. The objective of a feedback controller design is to choose these elements to stabilize the plant $G(s)$ and to meet various performance specifications.

A usual design technique is to obtain from analog design methods a compensator $K(s)$ and from here "translate" this rational transfer function in the s variable to a z transform $D(z)$. A prefilter $F(s)$ is selected such that its cutoff frequency, which depends on T, will minimize the aliasing effect. Usually a zero-order hold $H_0(s)$ is chosen for simplicity.

Several methods can be used to obtain an approximation $D(z)$ from $K(s)$, the main ones being[1] pole-zero mapping, nth-order hold equivalence, bilinear transformation, and prewarped Tustin. Of these, the last method is preferred because it allows $D(z)$ to have the same gain and phase as $K(s)$ at an important frequency such as the crossover ω_c. None of these methods of determining $D(z)$ takes into account the prefilter $F(s)$ and the hold $H(s)$, which effectively contribute some extra phase lag at ω_c.

In Ref. 2, the optimal cone center for the sampled data compensator has been found, taking into account prefilter and hold. In Ref. 3 the approximation of $K(s)\hat{F}(s)$ is made by one of the aforementioned methods, where $\hat{F}(s)$ is a lead compensation, which takes into account the effect of $F(s)H(s)$ over some frequency range. This method deals with the problem in an effective but rather ad-hoc approach, the main drawback being the difficulty in choosing $\hat{F}(s)$ in each case.

A straightforward approach[3] also adopted states the nonlinear programming problem:

$$J = \min_x \| K(s) - H(s)D(x, e^{sT})F(s) \|_2 \tag{1}$$

where $\|\cdot\|_2$ denotes the integral along the $j\omega$ axis of the squared function, and x is a vector of parameters of the rational transfer function $D(z)$. The main problem is that there is no guarantee that the closed-loop sampled data system remains stable. Furthermore, it can be shown that a good match between the analog and sampled data compensators is not enough to obtain a good match between their performance characteristics such as their sensitivity or complementary sensitivity functions.

A better approach is to minimize a new cost function, which represents some performance characteristics as a function of the parameters of $D(z)$. This method has been adopted,[4] but with no guarantee on the stability of the sampled-data, closed-loop system. Also in selecting the coefficients of the pole and zero polynomials, the parameters to be optimized can cause numerical problems in large-order systems.

In this paper, an improvement to the preceding methods is given by guaranteeing closed-loop stability, approximating a performance cost function, and improving the numerical condition of the problem. The general formulation will be stated in Sec. II. Section III describes a solution that is suboptimal but computationally less expensive than the general approach stated in Sec. IV. Examples for both approaches are presented in Sec. V.

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II. Problem Formulation

The problem to be solved is the following:

$$\min_{x \in X} \| \mathfrak{F}[D(x, e^{sT})] \|_2 \tag{2}$$

being X the feasible set of parameters of $D(z)$, which guarantee closed-loop stability of the sampled data system. Function $\mathfrak{F}$ represents the difference between performance objectives (e.g., weighted sensitivity) of the analog and sampled data systems subject to the constraint of closed-loop stability of the latter. It is assumed that the analog closed-loop system is internally stable.

To develop both procedures, we will consider (without loss of generality) the output of the closed-loop system as the function to be matched, stating the problem as (see Fig. 1)

$$\min_{\text{stabilizing } D(z,x)} \left\| \frac{G(s)K(s)r(s)}{1+G(s)K(s)} - \frac{G(s)D(z)H(s)[F(s)r(s)]^T}{1+[G(s)H(s)F(s)]^T D(z)} \right\|_2 = \epsilon^* \tag{3}$$

where $[A(s)]^T$ is the Laplace transform of the time function $a(t)$ sampled every T s.

III. Simplified Approach

The objective is to state an equivalent problem that can give in a simple way, with the least amount of computation, a tight, upper bound to the minimum ϵ^* in Eq. (3). For simplicity the approximation $G(s)H(s)[F(s)r(s)]^T \simeq G(s)H(s)F(s)r(s)$ has been made. This can be justified by noting that the extra aliasing introduced by the sampled $[F(s)r(s)]^T$ is small above the Nyquist frequency $\omega_N = \pi/T$ because $H(s)G(s)$ rolls off at a much lower frequency for a reasonable choice of T. Now define the equivalent problem

$$\min_{\text{stabilizing } \tilde{K}(s)} \left\| \frac{G(s)K(s)r(s)}{1+G(s)K(s)} - \frac{\tilde{G}(s)\tilde{K}(s)r(s)}{1+\tilde{G}(s)\tilde{K}(s)} \right\|_2$$

$$= \min_{\text{stable rational } Q(s)} \| [T(s) - \tilde{G}(s)Q(s)]r(s) \|_2 \tag{4}$$

by parameterizing all stabilizing controllers by any stable $Q(s)$[5]; $T(s)$ is the complementary sensitivity function, considering $r(s)$ as a frequency dependent weight and with $\tilde{G}(s) = G(s)H(s)F(s)$.

This problem can be solved in general by using Wiener-Hopf techniques, but under certain restrictions an even simpler solution exists. The optimal transfer function $Q(s)$ solving Eq. (4) is $\hat{Q}(s) = K(s)[F(s)H(s)(1+G(s)K(s))]^{-1}$ when $F(s)$ and $H(s)$ are minimum phase. The prefilter $F(s)$ can always be made minimum phase for any reasonable design. The hold $H(s)$ instead should be approximated by a first or second-order Padé approximation, which will be rational and minimum phase.

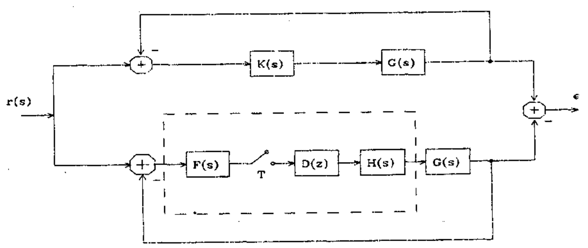

Fig. 1 Analog and sampled-data closed-loop system.

 J. GUIDANCE

The solution $\hat{Q}(s)$ will not be proper in general. In the next step we add ν extra poles in such a way as to make it proper, then $\hat{Q}_0(s,k) = \hat{Q}(s)/(s/k+1)^\nu$ will give us the optimal solution in the limit as $k \to \infty$. We are not interested in finding the value of k at this stage, so it will remain as a free parameter of the solution. Parameterizing Eq. (3) as we did in Eq. (4),

$$\min_{\text{stable rational } Q_d(z)} \| [T(s) - \tilde{G}(s) Q_d(z)] r(s) \|_2 = \epsilon^* \tag{5}$$

By making a Tustin transformation of $\hat{Q}_0(s,k)$, which preserves stability, we can replace this into Eq. (5) to obtain an upper bound on ϵ^*. This is

$$\hat{Q}_d(z,k) = \hat{Q}_0(s,k)\,|_{s=(z-1)/(z+1)} \tag{6}$$

then solve

$$\min_{k>0} \left\| T(s) - \hat{Q}_d(z,k)\tilde{G}(s) \right\|_2 = \epsilon \geq \epsilon^* \tag{7}$$

The optimal $\hat{Q}_d(z,k_0)$ provides a particular parameterization, which guarantees closed-loop stability of the sampled-data system from which we can obtain a suboptimal

$$D(z) = \frac{\hat{Q}_d(z,k_{\text{opt}})}{1 - \hat{Q}_d(z,k_{\text{opt}})[H(s)G(s)F(s)]^T} \tag{8}$$

The suboptimality of $D(z)$ comes from the fact that in Eq. (3) we are minimizing not over all stabilizing $D(z)$ but over a smaller set of controllers. It will be shown in the examples that the minimum ϵ obtained by this method is very close to the optimal ϵ^* with the advantage of simplicity (only a scalar minimization).

Although the parameterization of all stabilizing controllers by any stable $Q(s)$ is valid only for stable plants, the additional condition that the sensitivity function $(1-GQ)$ have zeros at the unstable poles of the plant, generalizes the procedure to include unstable plants as well.

IV. General Approach

This consists in solving directly Eq. (2) by a nonlinear inequality constrained optimization method. The inequality constraint comes from the fact that the poles of the closed-loop, sampled, data system must be inside the unit circle to guarantee stability. This is achieved by minimizing over the set X of parameters of $Q(z)$ in the following:

$$\left\| T(s) - Q(z,x)\tilde{G}(s) \right\|_2^2 \simeq \sum_{k=1}^{K} \Big| [T(i\omega_k) - \tilde{G}(i\omega_k) Q(e^{i\omega_k T},x)] r(i\omega_k) \Big|^2 \Delta\omega_k \tag{9}$$

with T and $\tilde{G}$ defined in Sec. III. In this case no restrictions need to be made for the hold $H(s)$ or for $[F(s)r(s)]^T$. This general approach is used when we do not obtain a satisfactory answer from the first approach. However the calculations involved are more expensive to perform.

It is very important to select a structure and a set of parameters for $Q(z,x)$ such that we do not restrict the pole and zero locations and at the same time to maintain a good numerical condition. The structure used is

$$Q(z,x) = x_L \frac{(x_{NQ}z+1)\Pi_{i=1}^{N}(z^2 - 2x_i z + x_i^2 - x_{i+N})}{(\tilde{x}_{MQ}z+1)\Pi_{j=1}^{M}(z^2 - 2\tilde{x}_j z + \tilde{x}_j^2 - \tilde{x}_{j+N})} \tag{10}$$

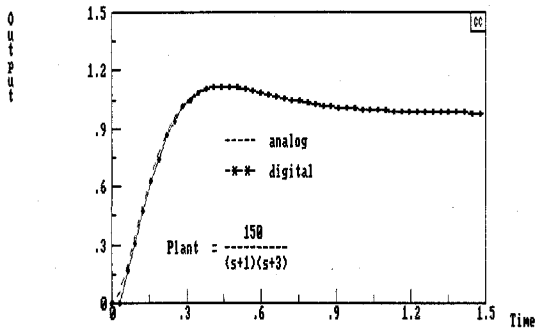

Fig. 2 Step response comparison for first example.

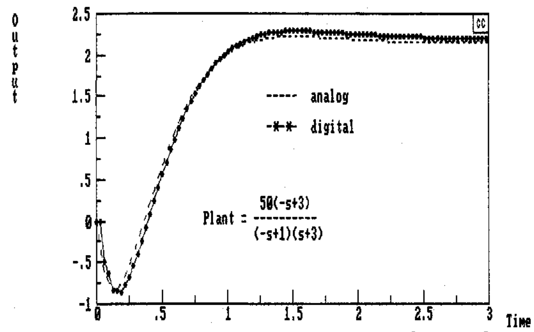

Fig. 3 Step response comparison for second example.

with

$$x^T = [x_1, \ldots, x_N, x_{N+1}, \ldots, x_{2N}, x_{NQ}, \tilde{x}_1, \ldots, \tilde{x}_M, \tilde{x}_{M+1}, \ldots, x_{2M}, x_{MQ}, x_L]$$

Taking $p(x) = z^2 - 2x_1 z + x_1^2 - x_2$ with the following possibilities,

If $x_2 < 0$, the roots are at $x_1 \pm i\sqrt{|x_2|}\ \epsilon C$ with $(x_1^2 + x_2) < 1$.

If $x_2 > 0$, the roots are at $x_1 \pm \sqrt{x_2}\ \epsilon R$.

If $x_2 = 0$, there is a double root at $x_1 \epsilon R$, and in these last two cases $|x_1 + \sqrt{x_2}| < 1$ and $|x_1 - \sqrt{x_2}| < 1$.

Taking $q(x) = (xz+1)$, the root is at $z_0 = -1/x$ with $|x| > 1$.

In this particular case, it is extremely complicated to obtain second-order information on the cost function, which means that we should restrict ourselves to the use of a gradient method. The optimization algorithm was selected among discrete Newton, Jacobian, or quasi-Newton methods. The latter seems to be the most convenient from the point of view of simplicity, speed, and storage. In particular, a Davidon-Fletcher-Powell algorithm[6] has been used with a Fibonacci's scalar search.

Another point to be considered is the resulting order of $Q(z)$. If we want to start from an initial point x_0 obtained from the first approach, then the order will be fixed and equal to $\mathcal{O}(Q) = \mathcal{O}(K) + \mathcal{O}(G) + \mathcal{O}(F) + 1 -$ (pole/zero cancellations in G and K).

Otherwise we can always specify an arbitrary order for $Q(z)$. For this last option, we must take into account that finally we should obtain a controller $D(z)$ from Eq. (8).

V. Examples

Two examples have been considered; the plants and their analog controllers are

$$G_1(s) = \frac{150}{(s+1)(s+3)}; \qquad K_1(s) = \frac{(s+3)^2}{(s+0.4)(s+22.5)} \tag{11}$$

$$G_2(s) = \frac{50(-s+3)}{(-s+1)(s+3)}; \qquad K_2(s) = \frac{-0.3(s+3)^2}{(s+2.864)(s+25.135)} \tag{12}$$

For both we use the second-order prefilter $F(s) = 2500/(s^2 + 70s + 2500)$ and a first-order Padé approximation for the zero-order hold $H(s)$. The sampling period is $T = \pi/100$ and the cutoff frequency $\omega_c = 6.9$ rad/s.

Using the approach in Sec. III, the optimum errors ϵ in Eq. (7) are in the order of 10^{-3} with the frequency range partitioned in 200 points. The digital controllers obtained had order five. Plots comparing the analog and sampled-data, closed-loop step responses are shown in Figs. 2 and 3, which present an improvement over all the methods mentioned in Sec. I.

For the general approach, the first example was considered using different starting points. It was observed that there is no significant reduction of the cost with respect to the first method, the initial being $\epsilon_0 = 0.0043$, which after ten iterations was reduced to $\epsilon = 0.0029$.

Acknowledgments

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State-Variable Models of Structures Having Rigid-Body Modes

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Introduction

THE equation of motion of structures is usually written in the form

$$M\ddot{x} + C\dot{x} + Kx = f_x \tag{1}$$

where M is the mass matrix, C is the damping matrix, and K is the stiffness matrix, all of order $(n \times n)$; and where x is the displacement vector and f_x the force vector, both of order $(n \times 1)$.[1] It will be assumed that M is positive definite, although the results could be generalized to the case of positive semidefinite M. If the structure has rigid-body freedom, K is singular. Equation (1) may be expanded to $2n$-order state-space form as follows:

$$A\dot{z} + Bz = f_z \tag{2}$$

where

$$A \equiv \begin{bmatrix} 0 & M \\ M & C \end{bmatrix}, \quad B \equiv \begin{bmatrix} -M & 0 \\ 0 & K \end{bmatrix}, \quad z \equiv \begin{Bmatrix} \dot{x} \\ x \end{Bmatrix}, \quad f_z \equiv \begin{Bmatrix} 0 \\ f_x \end{Bmatrix} \tag{3}$$

Let (λ_i, ϕ_i) be an (eigenvalue, right-eigenvector)-pair associated with the eigenproblem

$$(\lambda_i A + B)\phi_i = 0, \qquad i = 1,2,\ldots, 2n \tag{4}$$

This paper, which can be considered a supplement to Ref. 2, discusses some interesting features of the state-variable, rigid-body modes that arise when B is singular.

Generalized Eigenvectors, Jordan Form

An $n \times n$ matrix A that fails to have a linearly independent set of n eigenvectors is said to be defective. This may occur when A has a repeated eigenvalue. It is then not possible to transform A into diagonal form; i.e., there exists no Φ such that

$$A\Phi = \Phi\Lambda \tag{5}$$

where Λ is a diagonal matrix. But it is possible to find a linearly independent set of generalized eigenvectors, which transform A into the almost-diagonal Jordan form

$$AQ = QJ \tag{6}$$

Reference 3 defines these concepts and shows, for example, that when A has a repeated eigenvalue λ_2 of multiplicity three and an eigenvalue λ_3 of multiplicity two, the Jordan matrix will have the form

$$J = \left[\begin{array}{c|ccc|cc|c} \lambda_1 & & & & & & \\ \hline & \lambda_2 & 1 & 0 & & & \\ & 0 & \lambda_2 & 1 & & & \\ & 0 & 0 & \lambda_2 & & & \\ \hline & & & & \lambda_3 & 1 & \\ & & & & 0 & \lambda_3 & \\ \hline & & & & & & \lambda_4 \end{array}\right] \tag{7}$$

where the repeated eigenvalues lead to Jordan blocks having the eigenvalue on the diagonal and ones on the superdiagonal.

It will now be shown that systems which have rigid-body modes and which are described by the state-variable equation of the form of Eq. (2) require the use of generalized eigenvectors.

Undamped Systems with Rigid-Body Modes

Let an undamped system be described by the physical equation of motion

$$M\ddot{x} + Kx = 0 \tag{8}$$

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